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ON A FORM OF STEADY CAPILLARY-GRAVITATIONAL WAVES OF FINITE AMPLITUDE

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Ia. I. SEKERZH-ZEN'KOVICH
(Moscow)
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A problem concerning steady, capillary-gravitational waves of finite amplitude generated by pressure periodically distributed over the surface of an infinitely deep stream is considered. A rigorous solution of this problem is presented, with the surface pressure given, in the form of an infinite trigonometric series. In addition a particular case is investigated when the wavelength of the given pressure coincides with the length of the steady free wave corresponding to the specified flow velocity and constant pressure at the surface. The waves investigated here cease to exist when the periodic part of the pressure distributed over the surface vanishes identically and the flow becomes uniform. Such waves have been called induced [1]. An analogous problem for gravitational waves was investigated earlier [2] by the suthor. In addition, the author used the Levi-Civita method [3, 4] to reduce a similar problem for free capillary-gravitational waves, to a nonlinear differential equation.

In the present paper the problem is reduced to solving a certain nonlinear integral equation. The latter is discussed and its solution is constructed for any degree of approximation. The first three approximations are derived completely and an approximate equation describing the wave profile is given.

1. Statement of the problem and derivation of the basic integral equation. Consider a plane parallel steady motion of a perfect incompressible heavy fluid bounded only from above by a free surface at which the pressure is given by $p_0 = p_0' + p_0(x)$. Here $p_0' = \text{const}$ and $p_0(x)$ is a given periodic function of the horizontal coordinate x. The flow is assumed to move from left to right with constant velocity c, at an infinite depth. Since the pressure at the surface is a periodic function of x, the surface assumes the form of a periodic wave, stationary with respect to coordinates attached to a progressive wave moving with velocity c. The present paper shows that induced waves exist for any finite values of c.

Let the required wave and the pressure $p_0(x)$ both possess the same symmetry with respect to the vertical through the wave crest. The y-axis is chosen so as to coincide with the axis of symmetry, and is directed vertically upwards. The coordinate origin O is placed at the point of intersection of y-axis with the free surface and the x-axis is directed to the right.

The xy-plane of flow is taken as the plane of the complex variable $z = x + iy$. Conventional notation is used: φ is the velocity potential, ψ is the stream function, $w = \varphi + i\psi$ is the complex velocity potential, and U and V are the projections of the velocity vector $\mathbf{q}$ on the coordinate axes. We then have

$$dw/dz = -U + iV, \qquad U = -\partial\varphi / \partial x, \qquad V = -\partial\varphi/\partial y$$

The basic equation of the problem is obtained from the boundary condition by conformal mapping of the region occupied by a single wave and represented by a vertical semi-infinite strip bounded from above by a wavelike curve, onto a half-strip $0 \leqslant \varphi \leqslant c\lambda$, $0 \leqslant \psi \leqslant \infty$ in the w-plane, and then mapping the latter onto the interior of a unit circle with the center at the coordinate origin of the plane $u = u_1 + iu_2$. The wavelength λ is assumed to coincide with the period of the function $p_0(x)$.

The latter mapping is given by the known formula

$$w = \frac{\lambda c}{2\pi i} \ln u \tag{1.1}$$

the wave profile transforming into the circumference of the unit circle with a cut along the radius $\arg u = 0$.

Mapping of the circle $|u| \leqslant 1$ on a region in the z-plane occupied by a single wave is defined from the following relation:

$$\frac{dz}{du} = -\frac{\lambda}{2\pi i}\frac{f(u)}{u}, \qquad f(u) = 1 + \sum_{k=1}^{\infty} a_k u^k \tag{1.2}$$

The coefficients a_k are real, since the wave is symmetric about the y-axis and $a_0 = 1$ as the stream velocity at infinity is directed along the x-axis and equal to c.

Use of the function [1]

$$\omega(u) = \Phi + i\tau = -i \ln f(u) \tag{1.3}$$

together with (1.2) and (1.3) yields, for $u = e^{i\theta}$ (θ is the angle formed by the radius vector and the u_1-axis) a differential relation which, after separating the real and imaginary parts and integrating, gives the following parametric equation of the wave profile

$$x = -\frac{\lambda}{2\pi}\int_0^\theta e^{-\tau(\eta)} \cos\Phi(\eta)\, d\eta, \qquad y = -\frac{\lambda}{2\pi}\int_0^\theta e^{-\tau(\eta)} \sin\Phi(\eta)\, d\eta \tag{1.4}$$

Formulas (1.3), (1.2) and (1.1) imply that the function Φ is equal to the angle made by the velocity vector $\mathbf{q}$ with the x-axis everywhere in the stream and, that

$$q = |\mathbf{q}| = c \exp(\tau) \tag{1.5}$$

Passing to the boundary condition at the surface, we write the Bernoulli surface integral

$$p/\rho = C - gy - {}^1/_2\, q^2 \tag{1.6}$$

where C is a constant, g is the acceleration due to gravity and ρ the density. At the free surface any pressure difference is balanced by the normal component of the surface tension. For the latter we have, by the Laplace's rule,

$$p - p_0 = \pm \mu/R, \qquad p_0 = p_0' + p_0(x) \tag{1.7}$$

where p is the pressure exerted by the fluid, p_0 is the pressure from the direction of the free surface, μ is the capillary constant and R is the radius of curvature at the points on the surface. Expressing the curvature by $d\Phi / d\theta$ we obtain

$$p - p_0 = \frac{2\pi\mu}{\lambda c} q \frac{d\Phi}{d\theta} \tag{1.8}$$

Insertion of p from (1.8) into (1.6) now yields

$$\frac{d\Phi}{d\theta} = \nu \left[\delta e^{-\tau} - e^{\tau} - \frac{2\pi}{\lambda} \varkappa y e^{-\tau} - p_0^* (x) e^{-\tau}\right] \tag{1.9}$$

where

$$\nu = \frac{\lambda c^2 \rho}{4\pi\mu}, \quad \delta = \frac{2 (C\rho - p_0')}{\rho c^2}, \quad \varkappa = \frac{g\lambda}{\pi c^2}, \quad p_0^* (x) = \frac{2p_0 (x)}{\rho c^2} \tag{1.10}$$

The quantity y appearing in (1.9) is given by the second formula in (1.4). Separating the terms linear in Φ and τ in the right-hand side of (1.9) we obtain

$$\frac{d\Phi}{d\theta} = \nu \left\{\delta - 1 + (\delta + 1)\tau + \varkappa \int_0^\theta \Phi (\eta)\, d\eta - S (\theta)\ (1 - \tau) + F [\tau, \Phi, S, \delta]\right\} \tag{1.11}$$

where

$$F [\tau, \Phi, S, \delta] = \delta (e^{-\tau} - 1 + \tau) - (e^{\tau} - 1 - \tau) +$$
$$+ \varkappa e^{-\tau} \int_0^\theta [e^{-\tau (\eta)} \sin \Phi - \Phi (\eta)]\, d\eta - \varkappa \int_0^\theta \Phi (\eta)\, d\eta + \varkappa e^{-\tau} \int_0^\theta \Phi (\eta)\, d\eta -$$
$$- S (\theta) (e^{-\tau} - 1 + \tau)$$

Here it is assumed that

$$p_0^* (x) = S (\theta) = \sum_{n=1}^{\infty} \varepsilon^n d_n \cos n\theta \tag{1.12}$$

where ε is a small positive dimensionless parameter, d_n are given real numbers and the series

$$\sum_{n=1}^{\infty} \varepsilon^n d_n$$

converges in the circle $\varepsilon_0 > 0$, holds with the accuracy to within a constant included in p_0'. We note that in the initial problem $p_0^* (x)$ is a given periodic function of x. It can however be shown that solving our problem under the condition (1.12) is equivalent to specifying the series

$$p_0^* (x) = \sum_{n=1}^{\infty} \varepsilon^n c_n' \cos \frac{2\pi n}{\lambda} x, \qquad c_n' = \sum_{m=0}^{\infty} \varepsilon^m c_{mn}'$$

We can either assume here that the coefficients c_{0n}' are given and use them to determine d_n, or conversely we can obtain the coefficients c_{mn} $(m = 1, 2, \ldots)$ in terms of d_n (see Sect. 4, (4.3)). If it had been assumed that $d_n = d_{0n} + d_{1n}\varepsilon + d_{2n}\varepsilon^2 + \ldots$ (which had not been done here), then it would have been possible to assume that c_{mn} $(m = 1, 2, 3, \ldots)$ is also given, and to use them to determine d_{in} $(i = 1, 2, \ldots)$ or vice versa.

Equation (1.11) connects the functions $\tau (\theta)$ and $\Phi (\theta)$ on the circle $|u| = 1$, and the following known Dini relations are also valid for these functions

$$\Phi (\theta) = \int_0^{2\pi} K_0 (\eta, \theta) \frac{d\tau}{d\eta} d\eta, \quad K_0 (\eta, \theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\eta \sin n\theta}{n} \tag{1.13}$$

$$\tau (\theta) = -\int_0^{2\pi} K (\eta, \theta) \frac{d\Phi}{d\eta} d\eta, \quad K (\eta, \theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\eta \cos n\theta}{n}$$

Let us transform the components of (1.11) by applying to them (1.13) and integrating by parts. Collecting now the terms containing the same integrand function $d\Phi/d\eta$ and different kernels, we obtain

$$K(\eta, \theta) \text{ and } K_2(\eta, \theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\eta \cos n\theta}{n^2}$$

where $K_2(\eta, \theta)$ denotes the first iteration of the kernel $K(\eta, \theta)$.

The constants ν and $\varkappa$ in (1.11) are assumed known and δ is found from the condition of periodicity

$$\Phi(\theta + 2\pi) = \Phi(\theta)$$

Since $S(\theta)$ is given by (1.12), the solution of (1.11) and consequently δ, will depend on ε. Let us make the substitution

$$\delta = \delta_0 + \delta'(\varepsilon) \tag{1.14}$$

in (1.11). We then find from the condition of periodicity as $\varepsilon \to 0$ and from the fact that the quantity $\delta'(\varepsilon)$ and the solution both tend to zero, that $\delta_0 = 1$.

After the necessary transformations and with (1.14) taken into account, Eq. (1.11) assumes its final form

$$\zeta(\theta) = \nu \left\{ \int_0^{2\pi} K^*(\eta, \theta)\zeta(\eta)\,d\eta + \delta'(\varepsilon) + \delta'(\varepsilon) \int_0^{2\pi} K(\eta, \theta)\zeta(\eta)\,d\eta + \right.$$

$$+ \varkappa \int_0^{2\pi} K_2(\eta, \theta)\zeta(\eta)\,d\eta - S(\theta)\left[1 + \int_0^{2\pi} K(\eta, \theta)\zeta(\eta)\,d\eta\right] +$$

$$\left. + F[\tau, \Phi, S, 1 + \delta'(\varepsilon)] \right\} \tag{1.15}$$

$$\zeta(\theta) = \frac{d\Phi}{d\theta}, \quad K^*(\eta, \theta) = \sum_{n=1}^{\infty} \frac{\varphi_n(\eta)\varphi_n(\theta)}{\nu_n}, \quad \nu_n = \frac{n^2}{2n - \varkappa}, \quad \varphi_n(\theta) = \frac{\cos n\theta}{\sqrt{\pi}} \tag{1.16}$$

where ν_n denotes the eigenvalues and $\varphi_n(\theta)$ the eigenfunctions of the kernel $K^*(\eta, \theta)$.

The assumption that the function $\tau(\theta)$ in the expression for F is taken from (1.13) and

$$\Phi(\theta) = \int_0^{\theta} \frac{d\Phi}{d\eta}\,d\eta$$

implies that (1.15) represents a nonlinear integral equation for $\zeta(\theta) = d\Phi/d\theta$.

The condition of periodicity of $\Phi(\theta)$ gives

$$\delta'(\varepsilon) = -\varkappa \int_0^{2\pi} K_2(\eta, \theta)\zeta(\eta)\,d\eta + \frac{1}{2\pi} \int_0^{2\pi} \left\{ S(\theta)\left[1 + \int_0^{2\pi} K(\eta, \theta)\zeta(\eta)\,d\eta\right] - \right.$$

$$\left. - F[\tau, \Phi, S, 1 + \delta'(\varepsilon)] \right\} d\theta \tag{1.17}$$

Thus the problem has been reduced to the determination of the function $\zeta(\theta, \varepsilon)$ and the constant $\delta'(\varepsilon)$ from (1.15) and (1.17). Furthermore $\tau(\theta, \varepsilon)$ is obtained from (1.13) and

$$\Phi(\theta, \varepsilon) = \int_0^{\theta} \zeta(\eta, \varepsilon)\,d\eta \tag{1.18}$$

In solving these equations two cases may be considered: the case when $\nu \neq \nu_n$ and the case when $\nu = \nu_n$.

In the first case the solution $\zeta(\theta, \varepsilon)$ and the constant $\delta'(\varepsilon)$ are constructed in the form of series in integral powers of ε.

In the second case the solution appears as a power series in $\varepsilon^{1/3}$.

In both cases linear Fredholm integral equations of the second kind with the kernel $K^*(\eta, \theta)$ and the parameter ν are obtained for the coefficients of the expansion for $\zeta(\theta, \varepsilon)$. One linear equation is obtained for each coefficient of the expansion for $\delta'(\varepsilon)$. In the next section the equations defining the first coefficients of these expansions for the case $\nu = \nu_n$ are studied.

2. Solution of the linear problem for the case $\nu = \nu_n$ and the investigation of the kernel of the integral equation (1.15). Constructing the solution of (1.15) and (1.17) in the form of power series in $\varepsilon^{1/3}$, we arrive at the following system of equations defining the first coefficients of these expansions

$$\zeta_1(\theta) = \nu\left[\int_0^{2\pi} K^*(\eta, \theta)\,\zeta_1(\eta)\,d\eta + \delta_1 + \varkappa\int_0^{2\pi} K_2(\eta, 0)\,\zeta_1(\eta)\,d\eta\right] \qquad (2.1)$$

$$\delta_1 = -\varkappa\int_0^{2\pi} K_2(\eta, 0)\,\zeta_1(\eta)\,d\eta \qquad (2.2)$$

The same system of equations is obtained by assuming that $S(\theta) \equiv 0$ (free wave) in (1.15) and (1.17) and retaining linear terms.

Eliminating δ_1 from (2.2) and (2.1) and dropping the subscript, we obtain

$$\zeta(\theta) = \nu\int_0^{2\pi} K^*(\eta, \theta)\,\zeta(\eta)\,d\eta \qquad (2.3)$$

Since this is a linear homogeneous Fredholm equation of the second kind, by the Second Fredholm Theorem this has a nonzero solution only when $\nu = \nu_n$ (ν_n is the eigenvalue of the kernel $K^*(\eta, \theta)$. On the other hand by (1.10), the parameter $\nu > 0$ while ν_n, according to (1.16), depends on n and $\varkappa$. Since the parameter $\varkappa$ is assumed fixed, in order to arrive at a solution the dependence of ν_n on n must be investigated at a fixed value of $\varkappa = \varkappa_0$.

We now define the extremal value of ν_n. Differentiating with respect to n, we have

$$\frac{d\nu_n}{dn} = \frac{2n(n - \varkappa_0)}{(2n - \varkappa_0)^2}$$

This implies that ν_n reaches its minimum value at

$$n = \varkappa_0 \qquad (2.4)$$

At $2n = \varkappa_0$ the curve $\nu_n(n, \varkappa_0)$ has a vertical asymptote. Function $\nu_n < 0$ for $0 < n < {}^1/_2\,\varkappa_0$ and $\nu_n > 0$ for $n > {}^1/_2\,\varkappa_0$.

In addition, the following property of ν_n is established: for any two positive integers m and l such a $\varkappa_0$ can be chosen, that

$$\nu_m = \nu_l \qquad (2.5)$$

Indeed, inserting ν_m and ν_l into (2.5) we obtain a relation which, when solved for $\varkappa_0$, yields

$$\varkappa_0 = \frac{2ml}{m+l} \tag{2.6}$$

This particular value $\varkappa_0$ will have a corresponding eigenvalue $\nu_m = \nu_l$ associated with two linearly independent eigenfunctions $\varphi_m(\theta)$ and $\varphi_l(\theta)$, i.e. the eigenvalue will be double.

Thus inspection of $\nu_n = \nu_n(n, \varkappa_0)$ shows for a fixed $\varkappa_0$ the eigenvalue index $n > {}^1/_2\,\varkappa_0$ must be taken. With such $\nu = \nu_n$ Eq. (2.3) will have one or two linearly independent solutions, and only a finite number of multiple eigenvalues will exist. This can easily be confirmed by constructing a graph of ν_n versus n for a fixed $\varkappa = \varkappa_0$.

Assuming n fixed, let us now see how ν and $\varkappa$ are related when (2.3) has a nonzero solution. Setting $\nu = \nu_n$ we have

$$\nu = \frac{n^2}{2n - \varkappa} \tag{2.7}$$

which can be written in the form

$$\frac{1}{\nu} = \frac{1}{n^2}(2n - \varkappa) \tag{2.8}$$

Inserting into the latter the values for ν and $\varkappa$ given by (1.10) we obtain the known hyperbolic relation connecting c^2 and λ

$$c^2 = \frac{2\pi\mu}{\lambda\rho} n + \frac{g\lambda}{2\pi n}$$

This hyperbole has a vertical asymptote $\lambda = 0$ and an oblique asymptote $c^2 = g\lambda / 2\pi n$. In the first quadrant we find $c^2_{\min}$ corresponding to $\lambda = \lambda_*$ and we have

$$\lambda_* = 2\pi n \sqrt{\mu / \rho g}, \quad c^2_{\min} = 2\sqrt{\mu g / \rho}$$

The corresponding value of $\varkappa$ can now be called critical and denoted by $\varkappa_*$. From (1.10) we now find that

$$\varkappa_* = n \tag{2.9}$$

The hyperbolic branch corresponding to the values $0 < \lambda < \lambda_*$ resembles the branch $\lambda c^2 = 2\pi\mu n / \rho$ corresponding to pure capillary waves. When $\lambda > \lambda_*$, the values of c^2 increase together with λ and approach the asymptotic values $c^2 = g\lambda / 2\pi n$ corresponding to pure gravitational waves.

For this reason the waves considered split naturally into two types. The waves of the first type corresponding to $0 < \lambda < \lambda_*$ or to $0 < \varkappa < \varkappa_* = n$, and called capillary-gravitational, and the waves of the second type corresponding to $\lambda > \lambda_*$ or $\varkappa_* < \varkappa < 2n$ called gravitational-capillary.

Relation (2.8) makes it possible to include a special case of $2n - \varkappa = 0$. Indeed, (2.8) and (1.10) yield

$$\frac{2n - \varkappa}{n^2} = \frac{1}{\nu} = \frac{4\pi\mu}{\lambda c^2 \rho}$$

From this it follows that if $2n - \varkappa = 0$, $1 / \nu = 0$ and it must be assumed that $\mu = 0$. Consequently, (2.3) is replaced by an equation for gravitational waves without surface tension.

To confirm this we must return to the initial equation (1.11). Assume in this equation $S(\theta) \equiv 0$ retaining the linear terms only, dividing both sides of the resulting expression by ν and substituting $1 / \nu = 0$, we obtain

$$\delta - 1 - (\delta + 1)\tau + \varkappa \int_0^\theta \Phi(\eta)\, d\eta = 0 \tag{2.10}$$

Application of the first formula of (1.13) and integration by parts now yields

$$\int_0^\theta \Phi(\eta)\,d\eta = -\int_0^{2\pi} K(\eta, 0)\,\tau(\eta)\,d\eta + \int_0^{2\pi} K(\eta, \theta)\,\tau(\eta)\,d\eta \tag{2.11}$$

By (2.11), Eq. (2.10) becomes

$$\delta - 1 - (\delta + 1)\,\tau + \varkappa\left[\int_0^{2\pi} K(\eta, \theta)\,\tau(\eta)\,d\eta - \int_0^{2\pi} K(\eta, 0)\,\tau(\eta)\,d\eta\right] = 0$$

from which, setting

$$\delta = 1 + \varkappa\int_0^{2\pi} K(\eta, 0)\,\tau(\eta)\,d\eta$$

and retaining only the linear terms, we obtain

$$-2\tau(\theta) + \varkappa\int_0^{2\pi} K(\eta, \theta)\,\tau(\eta)\,d\eta = 0 \tag{2.12}$$

The eigenvalues of the kernel $K(\eta, \theta)$ are equal to $\mu_n = n$. Since the condition $\varkappa = 2n$ implies that $\varkappa = 2\mu_n$, Eq. (2.12) becomes the following homogeneous linear integral equation:

$$\tau(\theta) = \mu_n\int_0^{2\pi} K(\eta, \theta)\,\tau(\eta)\,d\eta \tag{2.13}$$

Since μ_n are the eigenvalues of $K(\eta, \theta)$, the above equation for free gravitational waves has a solution at any positive integer n.

A different approach is also feasible. Differentiation of (2.10) with respect to θ yields

$$(\delta + 1)\frac{d\tau}{d\theta} = \varkappa\Phi(\theta), \qquad \text{or} \qquad \frac{d\tau}{d\theta} = \frac{\varkappa}{\delta + 1}\Phi(\theta)$$

which, when inserted into the first formula of (1.13), gives

$$\Phi(\theta) = \frac{\varkappa}{\delta + 1}\int_0^{2\pi} K_0(\eta, \theta)\,\Phi(\eta)\,d\eta \tag{2.14}$$

The eigenvalues of the kernel $K_0(\eta, \theta)$ are $\mu_n = n$. Since here $\varkappa = 2n$, it follows that $\varkappa = 2\mu_n$. Substituting $\delta = 1$ and $\varkappa = 2\mu_n$ in Eq. (2.12) we arrive at the following known integral equation for free gravitational waves:

$$\Phi(\theta) = \mu_n\int_0^{2\pi} K_0(\eta, \theta)\,\Phi(\eta)\,d\eta \tag{2.15}$$

Finally we consider the case $\varkappa = 0$. For this case (2.7) yields $2\nu = n$. But $2\nu = \nu_n = n$ are the eigenvalues of the kernel $K(\eta, \theta)$. On the other hand, when $\varkappa = 0$, Eq. (2.3) becomes

$$\zeta(\theta) = 2\nu\int_0^{2\pi} K(\eta, \theta)\,\zeta(\eta)\,d\eta \tag{2.16}$$

The latter represents an integral equation for free capillary waves. Since $2\nu = \nu_n = n$, the equation has a solution for any positive integer n.

The results of our investigation of the linear problem can be stated as the following theorems.

Theorem 2.1. Let

$$\frac{1}{\nu} = \frac{1}{n^2}(2n - \varkappa)$$

where n is a fixed positive integer. Then Eq. (2.3) has a unique nontrivial solution

$$\zeta(\theta) = C_1\varphi_n(\theta) = \frac{C_1}{\sqrt{\pi}}\cos n\theta$$

for all values of $\varkappa$ within the interval $0 < \varkappa < 2n$.

If $\varkappa = \varkappa^{(m)} = 2mn\,/\,(m+n)$ (m is a positive integer),

$$\zeta(\theta) = C_2\varphi_m(\theta) = \frac{C_2}{\sqrt{\pi}}\cos m\theta$$

is also a particular solution linearly independent of $\varphi_n(\theta)$, and

$$\zeta(\theta) = C_1\varphi_n(\theta) + C_2\varphi_m(\theta) = \frac{C_1}{\sqrt{\pi}}\cos n\theta + \frac{C_2}{\sqrt{\pi}}\cos m\theta$$

in the general solution.

We shall call the values $\varkappa = \varkappa^{(m)}$ branching values. The waves corresponding to these values and defined by the solution $\zeta(\theta)$ in the form of a sum of two harmonics, shall be called double waves and the value $\varkappa = \varkappa_* = n$ – critical value. The latter divides the straight line (2.8) on the parametric plane ($y = 1\,/\,\nu$; $x = \varkappa$) into two parts. Points on the first part correspond to the capillary-gravitational waves and the points on the second part to the gravitational-capillary waves. When $m < n$, a finite number of the points of bifurcation appears on the first part and an enumerable set of these points appears on the second part when $n < m < +\infty$.

Theorem 2.2. When $\varkappa = 0$, Eq. (2.3) is transformed into (2.16) for pure capillary waves. For fixed $2\nu = \nu_n = n$, Eq. (2.16) has a unique nontrivial solution $\zeta(\theta) = C_1/\sqrt{\pi}\cos n\theta$ where n is a positive integer.

Theorem 2.3. When $\varkappa = 2n$, it is necessary to set $1\,/\,\nu = 0$ (consequently $\mu = 0$). Then Eq. (2.3) becomes (2.13) or (2.15) for pure gravitational waves. When n is a fixed positive integer, these equations have unique nontrivial solutions which are, respectively

$$\tau(\theta) = \frac{C_1}{\sqrt{\pi}}\cos n\theta, \quad \Phi(\theta) = \frac{C_1}{\sqrt{\pi}}\sin n\theta$$

Theorem 2.4. Capillary-gravitational waves can be double waves; but no pure gravitational or pure capillary double waves exist.

The last theorem is a direct consequence of Theorems 2.1 – 2.3. It expresses the property which essentially differs the stationary capillary-gravitational waves from the stationary gravitational or stationary capillary waves.

The linear problem on stationary free capillary-gravitational waves has been studied by the author in [3, 4] where it was reduced to a solution of a differential equation in the complex domain.

3. Solution of the basic equations of the problem. It has already been stated at the end of Sect. 1, that two cases must be considered when $\zeta(\theta, \varepsilon)$ and $\delta'(\varepsilon)$ are obtained from the basic equations (1.15) and (1.17), i.e. the case when $\nu \neq \nu_n$ and the case when $\nu = \nu_n$. A method of constructing a solution will be shown for each case and the first three approximations obtained. In the second case the value $\nu = \nu_1$ is used as an example, the parameter $\varkappa$ being chosen so as to make the eigenvalue ν_1 simple and positive.

1°. The case $\nu \neq \nu_n$. As we already stated, in this case the solution is constructed in the form of series in integral powers of the parameter ε. A linear nonhomogeneous integral Fredholm equation of the second kind with parameter ν is obtained for each coefficient of the expansion of the function $\zeta(\theta, \varepsilon)$. The First Fredholm Theorem is used to solve all these equations consecutively. One linear equation is obtained for each coefficient of the expansion $\delta'(\varepsilon)$. This equation presents an explicit expression for the coefficient of the given approximation in terms of the quantities obtained in the previous approximations.

Expressions for $\zeta(\theta, \varepsilon)$ and $\delta'(\varepsilon)$ obtained using the first three approximations follow

$$\zeta(\theta, \varepsilon) = \varepsilon C_{11} \cos\theta + \varepsilon^2 C_{22} \cos 2\theta + \varepsilon^3 (C_{13} \cos\theta + C_{33} \cos 3\theta) \tag{3.1}$$

$$\delta'(\varepsilon) = -\varepsilon\varkappa C_{11} + \varepsilon^2 [{}^1/_4\varkappa (C_{11}^2 - C_{22}) + {}^1/_2 d_1 C_{11}] - \\ - \varepsilon^3\varkappa (C_{13} + {}^1/_9 C_{33} + {}^1/_{18} C_{11}^3 + {}^1/_6 C_{11} C_{22}) \tag{3.2}$$

$$C_{11} = \frac{\nu\nu_1 d_1}{\nu - \nu_1}, \qquad C_{22} = -\frac{\nu\nu_2}{\nu_2 - \nu}(d_2 + {}^1/_2 C_{11} d_1 + {}^3/_4 \varkappa C_{11}^2) \tag{3.3}$$

$$C_{13} = \frac{\nu\nu_1}{\nu - \nu_1} C_{13}^*, \qquad C_{33} = \frac{\nu\nu_3}{\nu_3 - \nu} C_{33}^*$$

Here C_{13}^* is a linear function of C_{11}^3, $C_{11}C_{22}$, $C_{22}d_1$, $C_{11}d_2$, $C_{11}^2 d_1$, while C_{33}^* is a linear function of C_{11}^3, $C_{11}C_{22}$, $C_{11}^2 d_1$, $C_{22}d_1$, $C_{11}d_2$, d_3.

2°. The case of $\nu = \nu_1$. Here a linear homogeneous Fredholm equation of the second kind, the value of the parameter being $\nu = \nu_1$, is obtained for the first coefficient of the expansion for $\zeta(\theta, \varepsilon)$ the latter constructed in the form of a series in powers of $\varepsilon^{1/3}$. The Second Fredholm Theorem is used to solve this equation. Equations for all the subsequent coefficients are the same, but remain nonhomogeneous even with $\nu = \nu_1$. The Third Fredholm Theorem is used to solve them. The coefficient appearing in the solution of the homogeneous, nth approximation equation is found from the condition of solvability of the ($n + 2$)-th approximation equation.

The coefficients of the expansion for $\delta'(\varepsilon)$ are obtained in a similar manner to the case when $\nu \neq \nu_n$.

Expressions for $\zeta(\theta, \varepsilon)$ and $\delta'(\varepsilon)$ obtained using the first three approximations follow

$$\zeta(\theta, \varepsilon) = \varepsilon^{1/3} C_{11} \cos\theta + \varepsilon^{2/3} (C_{12} \cos\theta + C_{22} \cos 2\theta) + \\ + \varepsilon (C_{13} \cos\theta + C_{23} \cos 2\theta + C_{33} \cos 3\theta) \tag{3.4}$$

where

$$C_{11} = d_1^{1/3}\alpha^{1/3}, \qquad \alpha = \frac{32(\nu_2 - \nu_1)}{8(\nu_2 - \nu_1) + 9\varkappa^2\nu_1\nu_2} \tag{3.5}$$

$$C_{22} = \frac{3}{4}\varkappa C_{11}^2 \frac{\nu_1\nu_2}{\nu_1 - \nu_2}, \qquad C_{12} = -\frac{\varkappa({}^1/_2 C_{11}^2 + 5C_{22})}{9[1 + \varkappa(1 + 7\nu_1\nu_2\varkappa / 8(\nu_1 - \nu_2))]}$$

$$C_{23} = \varkappa C_{11} C_{12} \frac{\nu_1\nu_2}{\nu_1 - \nu_2}, \qquad C_{33} = \frac{\nu_1\nu_3}{\nu_3 - \nu_1} C_{33}^*$$

Here C_{33}^* is a linear function of C_{11}^3 and $C_{11}C_{22}$; C_{13} is analogous to C_{12} but with different coefficients.

It should be noted that in both cases $\tau(\theta, \varepsilon)$ is obtained from (1.13), and $\Phi(\theta, \varepsilon)$ from (1.18).

4. Determination of the wave profile. The wave profile is defined in the parametric form $x(\theta, \varepsilon)$ and $y(\theta, \varepsilon)$ from the relations (1.4) in which $\Phi(\theta, \varepsilon)$ and $\tau(\theta, \varepsilon)$ obtained should be inserted. Eliminating θ from parameteric equation, we then obtain the equation for the profile in the form $y = y(x, \varepsilon)$.

Equations of the wave profile accurate to the third order terms are given below, setting $k = 2\pi / \lambda$.

For the case $\nu \neq \nu_n$ we have

$$y(x, \varepsilon) = k^{-1} \{\varepsilon C_{11} (\cos kx - 1) + {}^1/_4\, \varepsilon^2 (C_{11}{}^2 - C_{22}) (1 - \cos 2kx) + + {}^1/_6\, \varepsilon^3 [(6C_{13} + {}^9/_4 C_{11}C_{22}) (\cos kx - 1) + ({}^1/_3 C_{11}{}^3 - {}^5/_4 C_{11}C_{22} + + {}^2/_3\, C_{33})(\cos 3kx - 1)]\} \tag{4.1}$$

where the coefficients C_{ij} are given by (3.3), while for the case $\nu = \nu_1$ we have

$$y(x, \varepsilon) = k^{-1} \{\varepsilon^{1/3} C_{11} (\cos kx - 1) + {}^1/_2 \varepsilon^{2/3} [2C_{12} (\cos kx - 1) + + {}^1/_2 (C_{11}{}^2 - C_{22}) (1 - \cos 2kx)] + {}^1/_6\, \varepsilon\, [(6\, C_{13} + {}^9/_4\, C_{11}C_{22}) \times \times (\cos kx - 1) + 3 (C_{11}C_{12} - {}^1/_2 C_{23}) (1 - \cos 2kx) + + ({}^1/_6 C_{11}{}^3 - {}^5/_4 C_{11}C_{22} + {}^2/_3\, C_{33}) (\cos 3kx - 1)]\} \tag{4.2}$$

where the coefficients C_{ij} are given by (3.5).

Note 4.1. The conditions adopted for this problem place the coordinate origin at the wave crest. Therefore when x is almost zero, y must be negative. From (4.1) and (4.2) it follows that the latter will be true only when $C_{11} > 0$. Assuming that $\nu_1 < \nu < \nu_2$ (in the case of $\nu \neq \nu_n$) and by virtue of (3.3) and (3.5) we may conclude that it is necessary to have $d_1 > 0$.

Note 4.2. Eliminating θ from (1.12) and the parametric equation of the profile we obtain an expression for the pressure given at the surface, as a function of x. For $\nu \neq \nu_n$ it has the form

$$p_0{}^* (x) = \varepsilon d_1 \cos kx + \varepsilon^2 (d_2 - d_1 C_{11}) \cos 2kx + + {}^1/_6\, \varepsilon^3 [(6d_2 C_{11} + {}^3/_4 d_1 (2C_{11}{}^2 - C_{22})) \cos kx + (6d_3 - 6d_2 C_{11} + + {}^3/_4 d_1 (C_{22} - 2C_{11}{}^2)) \cos 3kx] \tag{4.3}$$

When $\nu = \nu_1$, the formula obtained using the same approximation will be similar to the above, but containing only a single term. Formula (4.3) verifies the statement concerning the expression (1.12) made in Sect. 1.

Note 4.3. When $\nu = \nu_n$, we have the particular case mentioned in the beginning of this paper. Indeed, when $\nu = \nu_n$, formulas (1.10) and (1.16) yield an approximate linear expression (see formula following from (2.8)) connecting c and λ in the case in question.

5. Existence and uniqueness of the solution of the problem. Applying the Liapunov-Schmidt methods and their consequent development [5] we can establish the following theorems.

Theorem 5.1. When $\nu \neq \nu_n$, the system of equations (1.15) and (1.17) has a unique solution $\zeta(\theta, \varepsilon)$ and $\delta'(\varepsilon)$ ($\delta'(\varepsilon) = \delta(\varepsilon) - 1$) small in ε and continuous in θ ($0 \leqslant \theta \leqslant 2\pi$). This solution is an analytic function in ε for small $|\varepsilon| < \varepsilon_0$.

Theorem 5.2. When $\nu = \nu_1$, the system of equations (1.15) and (1.17) has a unique solution $\zeta(\theta, \varepsilon)$ and $\delta'(\varepsilon)$ small in ε and continuous in θ ($0 \leqslant \theta \leqslant 2\pi$)

This solution can be represented by a series in powers of $\varepsilon^{1/3}$ converging uniformly and absolutely for small $|\varepsilon| < \varepsilon_0$.

The proof of these theorems is not included here. It is only noted that the procedure is similar to that used in [6, 7].

The above theorems imply that the series for $\Phi(\theta, \varepsilon)$ and $\tau(\theta, \varepsilon)$ converge uniformly and absolutely. Convergence of the series in powers of ε and $\varepsilon^{1/3}$ (in the case of $\nu = \nu_1$) for integrand functions in (1.4) follows from the general theorems of analysis on substitution of one series into the other. The general theorems of analysis also yield the proof of convergence of the series (4.1), (4.2) and (4.3).

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METHODS OF MECHANICS OF A CONTINUOUS MEDIUM FOR THE DESCRIPTION OF MULTIPHASE MIXTURES

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R. I. NIGMATULIN
(Moscow)

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The principal assumptions in the construction of a general multivelocity model of a continuous multiphase medium are examined and the fundamental equations (for mass, momentum and energy) of mechanics are obtained for each phase in the heterogeneous mixture. On the basis of these equations a closed system is proposed which describes the motion of a dispersed mixture of two compressible phases in the presence of phase changes. Energy transitions in phase transformations are analyzed. The fundamental relationships on the surface of the discontinuity are derived. Proceeding from the assumption of